Dot Product

Goal: Define an algebraic operation to capture the geometric relationship between two vectors.

Def^n: The dot product of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $\vec{u} \cdot \vec{v} := u_1v_1 + u_2v_2 + u_3v_3$.

The dot product has many useful algebraic properties.

Prop: If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are vectors and $a \in \mathbb{R}$, then...

⓪ (zero) $\vec{0} \cdot \vec{u} = 0u_1 + 0u_2 + 0u_3 = 0$

① (commutativity) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \vec{v} \cdot \vec{u}$

② (scalar factor) $(a\vec{u}) \cdot \vec{v} = a(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (a\vec{v})$.

$(a\vec{u}) \cdot \vec{v} = \langle au_1, au_2, au_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = au_1v_1 + au_2v_2 + au_3v_3 = a(\vec{u} \cdot \vec{v})$

Similar computation $\rightsquigarrow$ $\vec{u} \cdot (a\vec{v}) = a(\vec{u} \cdot \vec{v})$

③ (distribution) $\vec{u} \cdot (\vec{v} + \vec{w}) =$

$\vec{u} \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$

$= (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$

④ (norm) $\vec{v} \cdot \vec{v} = |\vec{v}|^2$

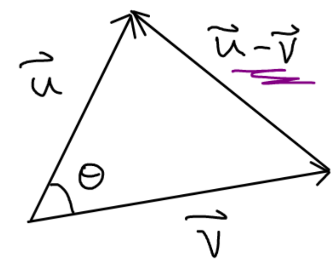
$\vec{v} \cdot \vec{v} = v_1v_1 + v_2v_2 + v_3v_3 = v_1^2 + v_2^2 + v_3^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2$

Now we relate the dot product to simple geometry.

Prop: If vectors $\vec{u}$ and $\vec{v}$ are at angle θ, then

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta).$$

pf: Consider the triangle formed by $\vec{u}$ and $\vec{v}$. By the Law of Cosines we obtain



$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)$$

By the basic properties of the dot product we obtain

$$|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\,\vec{u}\cdot\vec{v} = |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u}\cdot\vec{v}$$

Hence we obtain the equality

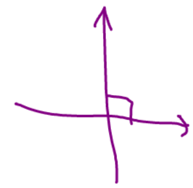
$$|\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v} = |\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta).$$

Finally, subtracting $|\vec{u}|^2 + |\vec{v}|^2$ from both sides and then dividing by -2 we obtain the desired result. ▨

NB: We can use this relationship to compute the angle between two vectors $\vec{u}$ and $\vec{v}$. :)

Defn: Vectors $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> when $\vec{u}\cdot\vec{v} = 0$.

Thus vectors $\vec{u}$ and $\vec{v}$ are orthogonal when



$$0 = \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$$

$$\iff \quad 0 = |\vec{u}| \quad \text{OR} \quad 0 = |\vec{v}| \quad \text{OR} \quad 0 = \cos(\theta)$$

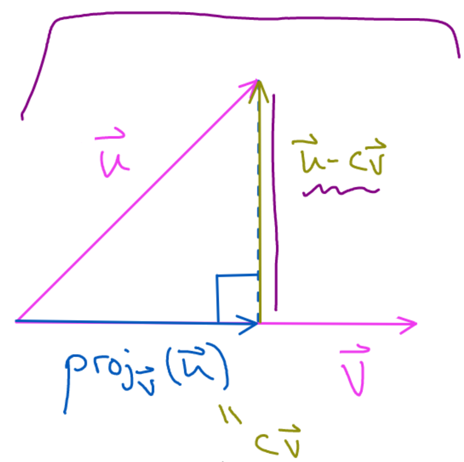
$$\iff \quad \vec{0} = \vec{u} \quad \text{OR} \quad \vec{0} = \vec{v} \quad \text{OR} \quad \theta = \arccos(0)$$

$$\iff \quad \left(\vec{u} = \vec{0} \quad \text{OR} \quad \vec{v} = \vec{0} \quad \text{OR} \quad \theta = \frac{\pi}{2}\right)$$

Thus ① the zero vector is orthogonal to every vector, and ② nonzero vectors are orthogonal when they're perpendicular.

ORTHOGONAL PROJECTION

Given two nonzero vectors, we can compute the "component of one in the direction of the other." We do so by projecting at a right angle.



This projection is a scalar multiple of $\vec{v}$. Thus

$\text{proj}_{\vec{v}}(\vec{u}) = c\vec{v}$ must satisfy the equation:

"vector projection of $\vec{u}$ onto $\vec{v}$"

$$\vec{v}\cdot(\vec{u} - c\vec{v}) = 0$$

$$\iff \vec{v}\cdot\vec{u} - c\,\vec{v}\cdot\vec{v} = 0 \iff \vec{u}\cdot\vec{v} = c\,\vec{v}\cdot\vec{v} = c|\vec{v}|^2 \iff c = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}$$

$\therefore$ We have $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\,\dfrac{\vec{v}}{|\vec{v}|}$

signed distance from line of $\vec{v}$ to head of $\vec{u}$

"scalar projection of $\vec{u}$ onto $\vec{v}$"

Ex: Compute the projection of $\vec{u}=\langle 3,-1,1\rangle$ onto $\vec{v}=\langle 4,7,-4\rangle$.

Sol: $\text{proj}_{\vec{v}}(\vec{u}) = \text{Comp}_{\vec{v}}(\vec{u})\,\frac{\vec{v}}{|\vec{v}|}$ (unit vector; scalar projection)

$\vec{u}\cdot\vec{v} = \langle 3,-1,1\rangle\cdot\langle 4,7,-4\rangle = 3\cdot 4 + -1\cdot 7 + 1\cdot -4 = 12-7-4 = 1$

$|\vec{v}| = \sqrt{4^2+7^2+(-4)^2} = \sqrt{16+49+16} = \sqrt{81} = 9$

$\text{Comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{1}{9}$ and $\frac{1}{|\vec{v}|}\vec{v} = \langle \frac{4}{9}, \frac{7}{9}, \frac{-4}{9}\rangle$

$\therefore\ \text{proj}_{\vec{v}}(\vec{u}) = \text{Comp}_{\vec{v}}(\vec{u})\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{9}\langle \frac{4}{9}, \frac{7}{9}, \frac{-4}{9}\rangle = \langle \frac{4}{81}, \frac{7}{81}, \frac{-4}{81}\rangle$. ///

In $\mathbb{R}^3$, we can use the dot product to compute angles to the standard axes. Let α, β, and γ denote the angles from a given vector $\vec{v}$ to the positive x-, y-, and z- axes respectively. Note that these axes are represented by the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.

If $\vec{v}=\langle x_0,y_0,z_0\rangle$, then we obtain

$$\vec{i}\cdot\vec{v} = \langle 1,0,0\rangle\cdot\langle x_0,y_0,z_0\rangle = x_0 \quad\text{and}$$

$$\vec{i}\cdot\vec{v} = |\vec{i}||\vec{v}|\cos(\alpha) = \sqrt{1^2+0^2+0^2}\cdot|\vec{v}|\cos(\alpha) = |\vec{v}|\cos(\alpha).$$

Hence when $\vec{v}\neq\vec{0}$ we obtain $\cos(\alpha) = \frac{x_0}{|\vec{v}|}$. Similarly we obtain $\cos(\beta) = \frac{y_0}{|\vec{v}|}$ and $\cos(\gamma) = \frac{z_0}{|\vec{v}|}$. Finally notice

$$\frac{\vec{v}}{|\vec{v}|} = \langle \frac{x_0}{|\vec{v}|}, \frac{y_0}{|\vec{v}|}, \frac{z_0}{|\vec{v}|}\rangle = \langle \cos(\alpha), \cos(\beta), \cos(\gamma)\rangle.$$

Examples

Ex: When is $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$?

Sol: $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ is always true.

So if $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$, then $|\vec{u}||\vec{v}| = |\vec{u}||\vec{v}|\cos(\theta)$.

If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, $\vec{u}\cdot\vec{v} = 0 = |\vec{u}||\vec{v}|$.

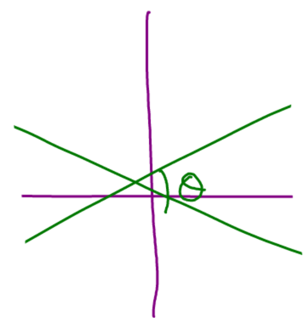
otherwise $1 = \cos(\theta)$, i.e. $\theta = \arccos(1) = 0$.

Hence $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$ if and only if either $\vec{u} = 0$ or $\vec{v} = 0$ or $\vec{u}$ and $\vec{v}$ have the same unit vector. //

Ex: Compute the accute angle between the lines

$$2x - 3y = 3 \quad \text{and} \quad 3x + y = 7.$$

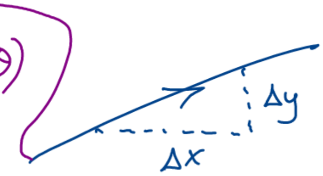
Sol: The direction of the lines are determined by:



$2x - 3y = 3 \iff y = \frac{-2}{-3}x + 3 \leadsto \vec{u} = \langle -3, -2\rangle$

$3x + y = 7 \iff y = \frac{-3}{1}x + 7 \leadsto \vec{v} = \langle 1, -3\rangle$

not to scale



$\therefore$ the desired angle satisfies $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

$\vec{u}\cdot\vec{v} = \langle -3, -2\rangle \cdot \langle 1, -3\rangle = -3 + 6 = 3$

$|\vec{u}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$

$|\vec{v}| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\therefore \; 3 = \sqrt{13}\sqrt{10}\cos(\theta) \iff \cos(\theta) = \frac{3}{\sqrt{130}} \iff \theta = \arccos\left(\frac{3}{\sqrt{130}}\right)$ //

Ex: Are the given vectors orthogonal, parallel, or neither?

ⓐ $\vec{u} = \langle -5, 4, -2 \rangle$ and $\vec{v} = \langle 3, 4, -1 \rangle$

$\vec{u} \cdot \vec{v} = \langle -5, 4, -2 \rangle \cdot \langle 3, 4, -1 \rangle = -15 + 16 + 2 = 3$ So not orthogonal.

$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{(-5)^2 + 4^2 + (-2)^2}} \langle -5, 4, -2 \rangle = \frac{1}{\sqrt{45}} \langle -5, 4, -2 \rangle = \langle -\frac{5}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, -\frac{2}{3\sqrt{5}} \rangle$

$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{3^2 + 4^2 + (-1)^2}} \langle 3, 4, -1 \rangle = \frac{1}{\sqrt{26}} \langle 3, 4, -1 \rangle = \langle \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}, -\frac{1}{\sqrt{26}} \rangle$

$\neq \pm$ each other, not parallel. $\therefore$ neither.

ⓑ $9\vec{i} - 6\vec{j} + 3\vec{k}$ and $-6\vec{i} + 4\vec{j} - 2\vec{k}$

$\vec{u} = \langle 9, -6, 3 \rangle$ $\qquad \vec{v} = \langle -6, 4, -2 \rangle$

$\vec{u} \cdot \vec{v} = -54 - 24 - 6 = -84 \neq 0$ so not orthogonal.

$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{9^2 + 6^2 + 3^2}} \langle 9, -6, 3 \rangle = \frac{1}{3\sqrt{3^2 + 2^2 + 1^2}} \langle 9, -6, 3 \rangle = \frac{1}{\sqrt{14}} \langle 3, -2, 1 \rangle$

$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{6^2 + 4^2 + 2^2}} \langle -6, 4, -2 \rangle = \frac{1}{2\sqrt{3^2 + 2^2 + 1^2}} \langle -6, 4, -2 \rangle = -\frac{1}{\sqrt{14}} \langle 3, -2, 1 \rangle$

Hence $\vec{u}$ is parallel to $\vec{v}$ ∎

Ex: Which of these expressions is meaningful? Why (not)?

ⓐ $(\vec{u} \cdot \vec{v}) \cdot \vec{w}$

scalar dot vect

Not meaningful! (can't dot vectors w/ scalars)

ⓑ $(\vec{u} \cdot \vec{v}) \vec{w}$

scalar vector

Is meaningful! (scalar multiple)

ⓒ $|\vec{u}| (\vec{v} \cdot \vec{w})$

scalar scalar

Is meaningful! (scalar product)

ⓓ $\vec{u} \cdot (\vec{v} + \vec{w})$

vector dot vector

Is meaningful! (vector dot vector)

ⓔ $\vec{u} \cdot \vec{v} + \vec{w}$

scalar plus vector

Not meaningful! (can't add scalars to vectors!)

ⓕ $|\vec{u}| \cdot (\vec{v} + \vec{w})$

scalar dot vector

Not meaningful! (can't dot vectors w/ scalars) ∎

Ex: Compute the angle between the given vectors.

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

$\iff \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \cos(\theta)$

$\iff \theta = \arccos\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$

(a) $\vec{u} = \langle -2, 5\rangle$, $\vec{v} = \langle 5, 12\rangle$

$\theta = \arccos\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\dfrac{-10+60}{\sqrt{4+25}\sqrt{25+144}}\right)$

$= \arccos\left(\dfrac{50}{\sqrt{29}\sqrt{169}}\right) = \arccos\left(\dfrac{50}{13\sqrt{29}}\right)$

(b) $\vec{u} = \langle -1, 3, 4\rangle$, $\vec{v} = \langle 5, 2, 1\rangle$

$\vec{u}\cdot\vec{v} = -5+6+4 = 5$

$|\vec{u}| = \sqrt{1+9+16} = \sqrt{26}$

$|\vec{v}| = \sqrt{25+4+1} = \sqrt{30}$

$\therefore \theta = \arccos\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\dfrac{5}{\sqrt{26}\sqrt{30}}\right)$

$= \arccos\left(\dfrac{5}{2\sqrt{13}\cdot\sqrt{15}}\right)$

(c) $\vec{u} = 4\vec{i} - 3\vec{j} + \vec{k}$, $\vec{v} = 2\vec{i} - \vec{k}$

$\vec{u} = \langle 4, -3, 1\rangle$ $\quad$ $\vec{v} = \langle 2, 0, -1\rangle$

$\vec{u}\cdot\vec{v} = 8 + 0 - 1 = 7$

$|\vec{u}| = \sqrt{16+9+1} = \sqrt{26}$

$|\vec{v}| = \sqrt{4+0+1} = \sqrt{5}$

$\therefore \theta = \arccos\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\dfrac{7}{\sqrt{130}}\right)$

$\blacksquare$

Ex: Compute the scalar and vector projections of $\underline{\vec{u} \text{ on } \vec{v}}_1$ and $\underline{\vec{v} \text{ on } \vec{u}}_2$ for $\vec{u} = \langle -1, 4, 8\rangle$ and $\vec{v} = \langle 12, 1, 2\rangle$.

$\text{comp}_{\vec{b}}(\vec{a}) = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$, $\quad \text{proj}_{\vec{b}}(\vec{a}) = \text{comp}_{\vec{b}}(\vec{a})\,\dfrac{\vec{b}}{|\vec{b}|}$

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u} = -12+4+16 = 8$

$|\vec{u}| = \sqrt{1+16+64} = \sqrt{81} = 9$

$|\vec{v}| = \sqrt{144+1+4} = \sqrt{149}$

$\text{proj}_{\vec{v}}(\vec{u}) \neq \text{proj}_{\vec{u}}(\vec{v})$

(1) $\text{comp}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \dfrac{8}{\sqrt{149}}$ and $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{8}{\sqrt{149}}\cdot\dfrac{1}{\sqrt{149}}\langle 12, 1, 2\rangle = \dfrac{8}{149}\langle 12, 1, 2\rangle$.

(2) $\text{comp}_{\vec{u}}(\vec{v}) = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|} = \dfrac{8}{9}$ and $\text{proj}_{\vec{u}}(\vec{v}) = \dfrac{8}{9}\cdot\dfrac{1}{9}\langle -1, 4, 8\rangle = \dfrac{8}{81}\langle -1, 4, 8\rangle$.

$\blacksquare$

Ex: Compute the direction cosines and angles of the vector $\vec{v} = \langle 6, 3, -2 \rangle$.

Sol: From the lecture $\frac{\vec{v}}{|\vec{v}|} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$.

$$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{36+9+4}} \langle 6, 3, -2 \rangle = \frac{1}{\sqrt{49}} \langle 6, 3, -2 \rangle = \left\langle \frac{6}{7}, \frac{3}{7}, -\frac{2}{7} \right\rangle$$

$$\therefore \underbrace{\begin{cases} \cos(\alpha) = \frac{6}{7} \\ \cos(\beta) = \frac{3}{7} \\ \cos(\gamma) = -\frac{2}{7} \end{cases}}_{\text{direction cosines}} \qquad \therefore \underbrace{\begin{cases} \alpha = \arccos\left(\frac{6}{7}\right) \\ \beta = \arccos\left(\frac{3}{7}\right) \\ \gamma = \arccos\left(-\frac{2}{7}\right) \end{cases}}_{\text{direction angles}}$$

$\square$

Ex: Show any two direction angles determine the third in $\mathbb{R}^3$.

Sol: We know $\frac{\vec{v}}{|\vec{v}|} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$, so

$$1^2 = \left| \frac{\vec{v}}{|\vec{v}|} \right|^2 = \left| \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle \right|^2$$

$$= \cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma)$$

$\therefore$ any two direction cosines determines the third!! $\square$

Ex: Compute the distance from point (x_0, y_0) to line $ax+by+c=0$.

Sol:

Prop: Let $\vec{u}$ and $\vec{v}$ be vectors. The vector $|\vec{v}|\vec{u} + |\vec{u}|\vec{v}$ bisects the angle between $\vec{u}$ and $\vec{v}$.

pf:

Prop (CAUCHY-SCHWARZ INEQUALITY): Every pair of vectors $\vec{u}, \vec{v}$ satisfies $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$.

pf:

Prop (TRIANGLE INEQUALITY): For every pair of vectors $\vec{u}, \vec{v}$ we have $|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$.

pf:

Prop (PARALLELOGRAM LAW): Every pair of vectors $\vec{u}, \vec{v}$ satisfies

$$|\vec{u}+\vec{v}|^2 + |\vec{u}-\vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2.$$

pf: